

Quasinormal modes for topologically massive black hole

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Abstract

We calculate quasinormal modes of a massive scalar perturbation on the topologically massive black hole (CS-BTZ black hole). The chiral point of $\mu\ell = 1$ corresponds to a newly extremal black hole. We show that there is no quasinormal modes at this point. Accordingly, we prove the unitarity of the CS-BTZ black hole at the chiral point.

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1 Introduction

The gravitational Chern-Simons terms in three-dimensional Einstein gravity produces a physically propagating massive graviton [1]. This theory with a negative cosmological constant $\Lambda = -1/\ell^2$ gives us the BTZ solution with mass m and angular momentum j as a trivial solution [2, 3]. However, there exists also the mixed solution like [4, 5]

$$M = m + \frac{j}{\mu\ell^2}, \quad J = j + \frac{m}{\mu}. \quad (1)$$

Hence, the horizon radius was shifted due to the presence of Chern-Simons term and thus the CS-BTZ black hole appeared as a new solution [6]. In this case, the authors found a newly extremal black hole at the chiral point of $\mu\ell = 1$, in addition to the extremal BTZ black hole at $\ell m = j$.

Recently, Sachs and Solodukhin have calculated quasinormal modes of a massive graviton on the non-rotating BTZ black hole background with the correspondence of $\mu\ell \leftrightarrow -m$ [8]. They have showed that quasinormal modes are absent for the case of $\mu\ell = 1$ with the central charges $c_L = 0$ and $c_R = 3\ell/G_3$ because of the zero right-moving conformal weight $h_R(\mu)|_{\mu=1/\ell} = 0$. This implies that the quasinormal mode of a massive graviton is always zero, since the massive graviton becomes a massless left-moving graviton, an unphysical propagation. However, we remind the reader that two horizons of CS-BTZ black hole becomes degenerate at the chiral point of $\mu\ell = 1$ [9]. This means that in order to see the effect of Chern-Simons term on the black hole, one needs to calculate quasinormal modes of a massive scalar on the CS-BTZ black hole background.

In this Letter, we compute quasinormal modes of a massive scalar perturbation on the CS-BTZ black hole. We show that there is no quasinormal modes at the chiral point. Accordingly, we prove the unitarity of the CS-BTZ black hole at the chiral point.

2 Scalar modes

We start with the topologically mass gravity in anti-de Sitter spacetimes [1]

$$I_{\text{TMG}} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left[R + \frac{2}{\ell^2} - \frac{1}{2\mu} \varepsilon^{\lambda\mu\nu} \Gamma^\rho_{\lambda\sigma} \left(\partial_\mu \Gamma^\sigma_{\nu\rho} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right) \right], \quad (2)$$

where ε is the tensor density defined by $\varepsilon/\sqrt{-g}$ with $\varepsilon^{012} = 1$. The $1/\mu$ -term is the first higher derivative correction in three dimensions because it is the third-order derivative.

Varying the this action leads to

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad (3)$$

where the Einstein tensor including the cosmological constant is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu} \quad (4)$$

and the Cotton tensor is

$$C_{\mu\nu} = \varepsilon_\mu^{\alpha\beta} \nabla_\alpha \left(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R \right). \quad (5)$$

The BTZ black hole solution is given by

$$ds_{BTZ}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi + N^\phi(r)dt \right)^2, \quad (6)$$

where the metric function f and the lapse function N^ϕ are

$$f(r) = -8G_3m + \frac{r^2}{\ell^2} + \frac{16G_3^2j^2}{r^2} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2r^2}, \quad (7)$$

$$N^\phi(r) = -\frac{4G_3J}{r^2} = -\frac{r_+r_-}{\ell r^2} \quad (8)$$

with two horizons

$$r_\pm = \ell \left[\sqrt{2G_3 \left(m + \frac{j}{\ell} \right)} \pm \sqrt{2G_3 \left(m - \frac{j}{\ell} \right)} \right]. \quad (9)$$

Here m and j are the mass and angular momentum of the BTZ black hole, respectively.

On the other hand, plugging Eq.(1) into Eq.(6) ($M \rightarrow m, J \rightarrow j$) leads to the CS-BTZ black hole solution [6] as

$$ds_{CS-BTZ}^2 = -\tilde{f}(r)dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2(d\phi + \tilde{N}^\phi(r)dt)^2, \quad (10)$$

where

$$\tilde{f}(r) = -8G_3 \left(m + \frac{j}{\mu\ell^2} \right) + \frac{r^2}{\ell^2} + \frac{16G_3^2(j + m/\mu)^2}{r^2} = \frac{(r^2 - \tilde{r}_+^2)(r^2 - \tilde{r}_-^2)}{\ell^2r^2} \quad (11)$$

$$\tilde{N}^\phi(r) = -\frac{4G_3(j + m/\mu)}{r^2} = -\frac{\tilde{r}_+\tilde{r}_-}{\ell r^2}. \quad (12)$$

Here we have two shifted horizons

$$\tilde{r}_\pm = \ell \left(\sqrt{1 + \frac{1}{\mu\ell}} \sqrt{2G_3 \left(m + \frac{j}{\ell} \right)} \pm \sqrt{1 - \frac{1}{\mu\ell}} \sqrt{2G_3 \left(m - \frac{j}{\ell} \right)} \right). \quad (13)$$

We observe that the degenerate horizon of $\tilde{r}_+ = \tilde{r}_- \equiv \tilde{r}_e$ appears at $\mu\ell = 1$ (chiral point) and $\ell m = j$. The former one is a newly extremal black hole due to the Chern-Simons term and the latter exists even for the absence of the Chern-Simons term.

In order to obtain quasinormal modes [7], we introduce the wave equation for a massive scalar field Ψ on the CS-BTZ black hole background of Eq.(10)

$$(\nabla_{\text{CS-BTZ}}^2 - m^2)\Psi(t, r, \phi) = 0, \quad (14)$$

where m is the mass of perturbed field with $m^2 \geq 0$. Considering the axially symmetric background, we parameterize the perturbed field as

$$\Psi(t, r, \phi) = e^{-i\omega t} e^{in\phi} \psi(r) \quad (15)$$

with angular number $n \in \mathbb{Z}$. Then, the wave equation can be written as

$$\tilde{f} \frac{d^2 \psi(r)}{dr^2} + \left(\frac{\tilde{f}}{r} + \frac{d\tilde{f}}{dr} \right) \frac{d\psi(r)}{dr} + \left(\frac{(\omega + \tilde{N}^\phi n)^2}{\tilde{f}} - \frac{n^2}{r^2} - m^2 \right) \psi(r) = 0. \quad (16)$$

Introducing a new variable $z = (r^2 - \tilde{r}_+^2)/(r^2 - \tilde{r}_-^2)$, the above equation can be transformed into

$$z(1-z) \frac{d^2 \psi(z)}{dz^2} + (1-z) \frac{d\psi(z)}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z} \right) \psi(z) = 0, \quad (17)$$

where

$$A = \frac{\ell^4}{4(\tilde{r}_+^2 - \tilde{r}_-^2)^2} \left(\omega \tilde{r}_+ - \frac{n}{\ell} \tilde{r}_- \right)^2 \equiv \alpha_+^2, \quad (18)$$

$$B = -\frac{\ell^4}{4(\tilde{r}_+^2 - \tilde{r}_-^2)^2} \left(\omega \tilde{r}_- - \frac{n}{\ell} \tilde{r}_+ \right)^2 \equiv -\alpha_-^2, \quad (19)$$

$$C = -\frac{m^2 \ell^2}{4}. \quad (20)$$

From now on, we assume $\alpha_\pm > 0$ to have a propagating wave. The general solution to Eq.(17) at the near horizon of $z \sim 0$ is given by the hypergeometric function ${}_2F_1$ with two constants C_1 and C_2 as

$$\psi(z) = z^{i\alpha_+} (1-z)^{(1-\Delta)/2} \left[C_1 {}_2F_1(a, b; c; z) + C_2 z^{1-c} {}_2F_1(a+1-c, b+1-c; 2-c; z) \right], \quad (21)$$

where the parameters Δ , a , b , and c take the forms

$$\Delta = \sqrt{1 + m^2 \ell^2}, \quad (22)$$

$$a = \frac{1}{2}(1 - \Delta) + i(\alpha_+ + \alpha_-), \quad (23)$$

$$b = \frac{1}{2}(1 - \Delta) + i(\alpha_+ - \alpha_-), \quad (24)$$

$$c = 1 + 2i\alpha_+. \quad (25)$$

We note the solution of the limiting case ($z \rightarrow 0$)

$$\psi_{z \rightarrow 0}(z) = C_1 z^{i\alpha_+} + C_2 z^{-i\alpha_+}. \quad (26)$$

Then, we obtain the corresponding solution $\Psi(t, r, \phi)$ as

$$\Psi_{z \rightarrow 0}(t, z, \phi) = e^{in\phi} \{C_1 e^{-i(\omega t - \alpha_+ \ln z)} + C_2 e^{-i(\omega t + \alpha_+ \ln z)}\}, \quad (27)$$

where the first term corresponds to an outgoing wave, while the second is an ingoing wave at the outer horizon. We note that the boundary condition for the quasinormal modes in asymptotically AdS spacetimes: ingoing mode (ingoing flux) at the outer horizon and no non-normalizable mode (zero flux) at infinity.

Since we have an ingoing wave at the horizon, it requires that $C_1 = 0$. Hence the solution at the near horizon is given by

$$\psi(z) = C_2 z^{-i\alpha_+} (1 - z)^{(1-\Delta)/2} {}_2F_1(a + 1 - c, b + 1 - c; 2 - c; z). \quad (28)$$

Now we are in a position to obtain the solution at infinity ($z = 1$), by applying the transformation formula for hypergeometric functions. The result is

$$\begin{aligned} \psi(z) = & C z^{-i\alpha_+} (1 - z)^{(1-\Delta)/2} \times \\ & \left\{ \frac{\Gamma(2 - c)\Gamma(c - a - b - 1)}{\Gamma(1 - a)\Gamma(1 - b)} {}_2F_1(a + 1 - c, b + 1 - c; a + b + 1 - c; 1 - z) \right. \\ & \left. + (1 - z)^{c-a-b} \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} {}_2F_1(1 - a, 1 - b; c - a - b + 1; 1 - z) \right\} \end{aligned} \quad (29)$$

where the first term corresponds to the non-normalizable mode and the second is the normalizable one. In order to obtain the normalizable term at infinity only, we require the conditions

$$1 - a = -k, \quad \text{or} \quad 1 - b = -k \quad (30)$$

with mode number $k = 0, 1, 2, \dots$. The overtones are defined as $k \neq 0$ if the quasinormal modes are present. From this condition, we find the two types of quasinormal modes for $\tilde{r}_+ \neq \tilde{r}_-$ as

$$\omega_L = \frac{n}{\ell} - 2i \left(\frac{\tilde{r}_+ - \tilde{r}_-}{\ell^2} \right) \left(k + \frac{1}{2} + \frac{\sqrt{1 + m^2 \ell^2}}{2} \right), \quad (31)$$

$$\omega_R = -\frac{n}{\ell} - 2i \left(\frac{\tilde{r}_+ + \tilde{r}_-}{\ell^2} \right) \left(k + \frac{1}{2} + \frac{\sqrt{1 + m^2 \ell^2}}{2} \right), \quad (32)$$

where ω_L describes the left-moving degrees of freedom and ω_R describes the right-moving degrees of freedom, in accordance with the CFT picture on the boundary at infinity [7]. Expressing these as the parameters of the CS-BTZ black holes leads to

$$\omega_L = \frac{n}{\ell} - \frac{4i}{\ell} \sqrt{1 - \frac{1}{\mu\ell}} \sqrt{2G_3 \left(m - \frac{j}{\ell}\right)} \left(k + \frac{1}{2} + \frac{\sqrt{1 + m^2\ell^2}}{2}\right) \quad (33)$$

$$\omega_R = -\frac{n}{\ell} - \frac{4i}{\ell} \sqrt{1 + \frac{1}{\mu\ell}} \sqrt{2G_3 \left(m + \frac{j}{\ell}\right)} \left(k + \frac{1}{2} + \frac{\sqrt{1 + m^2\ell^2}}{2}\right). \quad (34)$$

The normalizable solution at infinity is then given by

$$\psi_{z \rightarrow 1}(z) = C_2 z^{-i\alpha_+} (1-z)^{(1+\Delta)/2} \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} {}_2F_1(1-a, 1-b; c-a-b+1; 1-z). \quad (35)$$

We observe what happens as μ approaches the chiral point ($\mu\ell \rightarrow 1$). In this case, apparently, we have $\omega_L \rightarrow n/\ell - i0$ while $\omega_R \rightarrow -n/\ell - i4(\tilde{r}_e/\ell^2)(k + 1/2 + \Delta/2)$. This may be consistent with the picture of chiral gravity with central charges $c_L = 0$ and $c_R = 3l/G_3$. However, to obtain quasinormal modes with ω_R on the extremal black hole background, we have to make a further study because the coordinate of z is not appropriate for describing the extremal case of $\tilde{r}_+ = \tilde{r}_-$. Unfortunately, we have the same point of $z = 1$ at infinity of $r \rightarrow \infty$ and the extremal point of $\tilde{r}_+ = \tilde{r}_-$.

3 Scalar modes at the chiral point

For the extremal black holes, we have the horizon radius as

$$\tilde{r}_e = \ell \sqrt{1 + \frac{1}{\mu\ell}} \sqrt{2G_3 \left(m + \frac{j}{\ell}\right)}. \quad (36)$$

For $\mu\ell = 1$, we have $\tilde{r}_e = 2\ell\sqrt{G_3(m + j/\ell)}$, while for $m\ell = j$, we have $\tilde{r}_e = 2\ell\sqrt{1 + 1/\mu\ell}\sqrt{G_3m}$. In this section, we consider the first case only. To solve the wave equation (16) on the extremal CS-BTZ black hole background [10, 11], we introduce a new variable \tilde{z} as

$$\tilde{z} = -2i\Omega_- \frac{\tilde{r}_e^2}{r^2 - \tilde{r}_e^2}, \quad (37)$$

where

$$\Omega_{\pm} = \frac{\ell^2}{2\tilde{r}_e} \left(\omega_e \pm \frac{n}{\ell}\right). \quad (38)$$

Then, Eq.(16) becomes

$$\frac{d^2\psi}{d\tilde{z}^2} + \left\{ -\frac{1}{4} + \frac{i\Omega_+}{2} \frac{1}{\tilde{z}} - \frac{m^2\ell^2}{4} \frac{1}{\tilde{z}^2} \right\} \psi = 0, \quad (39)$$

which is the Whittaker's equation [12]. Its solution is given by

$$\psi(\tilde{z}) = \tilde{C}_1 M\left(\frac{i}{2}\Omega_+, \frac{\Delta}{2}, \tilde{z}\right) + \tilde{C}_2 W\left(\frac{i}{2}\Omega_+, \frac{\Delta}{2}, \tilde{z}\right), \quad (40)$$

where M is the Whittaker M-function and W is the Whittaker W-function. Here $\Delta = \sqrt{1 + m^2\ell^2}$. They are related to confluent hypergeometric functions F and U , respectively

$$M(a, b, \tilde{z}) = e^{-\tilde{z}/2} \tilde{z}^{1/2+b} F(1/2 + b - a, 1 + 2b, \tilde{z}), \quad (41)$$

$$W(a, b, \tilde{z}) = e^{-\tilde{z}/2} \tilde{z}^{1/2+b} U(1/2 + b - a, 1 + 2b, \tilde{z}). \quad (42)$$

For $\tilde{z} \rightarrow 0$, $W \sim \tilde{z}^{\frac{1-\Delta}{2}}$ diverges. Thus, we choose $\tilde{C}_2 = 0$ to have a normalizable mode at infinity. We obtain a final solution at infinity

$$\psi(\tilde{z}) = \tilde{C}_1 M\left(\frac{i}{2}\Omega_+, \frac{\Delta}{2}, \tilde{z}\right). \quad (43)$$

Now we can derive an explicit solution near the horizon at $\tilde{z} = \infty (r = \tilde{r}_e)$. For this purpose, we introduce an asymptotic expansion of the confluent hypergeometric function

$$F(a, c; \tilde{z}) \rightarrow \frac{\Gamma(c)}{\Gamma(c-a)} \tilde{z}^{-a} e^{\pm i\pi a} + \frac{\Gamma(c)}{\Gamma(a)} \tilde{z}^{a-c}, \quad (44)$$

where the upper sign being taken if $-\pi/2 < \arg(\tilde{z}) < 3\pi/2$ and the lower one if $-3\pi/2 < \arg(\tilde{z}) \leq -\pi/2$. Using this relation, we find the solution near the horizon as

$$\begin{aligned} \psi(\tilde{z}) &\simeq e^{-\tilde{z}/2} \tilde{z}^{-(1+\Delta)/2+i\Omega_+/2} \frac{\Gamma(1+\Delta)}{\Gamma(1/2 + \Delta/2 + i\Omega_+/2)} e^{\pm i\pi(1/2+\Delta/2-i\Omega_+/2)} \\ &+ e^{\tilde{z}/2} \tilde{z}^{(1+\Delta)/2-i\Omega_+/2} \frac{\Gamma(1+\Delta)}{\Gamma(1/2 + \Delta/2 - i\Omega_+/2)} \\ &\equiv \psi^{\text{in}}(\tilde{z}) + \psi^{\text{out}}(\tilde{z}). \end{aligned} \quad (45)$$

In order to obtain quasinormal modes, the wave function should be a purely incoming mode near the event horizon. This requirement could be accomplished by demanding $\psi^{\text{out}}(\tilde{z}) = 0$ as

$$\frac{1}{2} + \frac{\Delta}{2} - \frac{i}{2}\Omega_+ = -k, \quad (46)$$

for $k = 0, 1, 2, \dots$. Inverting this expression leads to the quasinormal frequencies

$$\tilde{\omega}_e = -\frac{n}{\ell} - 4i\frac{\tilde{r}_e}{\ell^2} \left(k + \frac{1}{2} + \frac{\sqrt{1 + m^2\ell^2}}{2} \right), \quad (47)$$

which is the same form recovered from ω_R in Eq.(32) with $\mu\ell = 1$. However, the condition of Eq.(46) leads in turn to the zero ingoing flux. This is because the flux expression

$$\mathcal{F}_{\text{in}}(\tilde{z} = \infty) \propto \frac{\Gamma(1 + \Delta)}{\Gamma(1/2 + \Delta/2 - i\Omega_+/2)} \frac{\Gamma(1 + \Delta)}{\Gamma(1/2 + \Delta/2 + i\Omega_+/2)} \quad (48)$$

leads to zero exactly when choosing $1/2 + \Delta/2 - i\Omega_+/2 = -k$. This implies that there is no room to accommodate quasinormal modes with $\omega_R = \tilde{\omega}_e$ of a massive scalar on the background of the extremal CS-BTZ black hole. The absence of quasinormal modes is consistent with the picture of a stable event horizon with thermodynamic properties $T_H = C_J = 0$, $S_{BH} = \pi\tilde{r}_e/2G_3$ of the extremal CS-BTZ black hole. This is so because the presence of quasinormal modes implies that a massive scalar wave is losing its energy continuously into the extremal event horizon.

Furthermore, the absence of ω_L implies that the extremal CS-BTZ black hole is chiral, in compared with $\omega_{L/R}$ in Eqs.(33) and (34) for the non-extremal CS-BTZ black holes.

4 Discussions

We study the wave equation for a massive scalar in three-dimensional CS-BTZ black hole spacetimes to understand the chiral point of $\mu\ell = 1$. Here we introduce two interesting black hole spacetimes: the CS-BTZ and extremal CS-BTZ. In the CS-BTZ case, one finds quasinormal modes. The presence of quasinormal modes means that it shows a leakage of information into the event horizon (dissipative object) and thus it signals a breakdown of the unitarity.

On the other hand, we do not find any quasinormal modes for the extremal CS-BTZ black hole. Instead, we find a real and continuous frequency ω_e for the extremal CS-BTZ case [10] which implies that the extremal CS-BTZ is a unitary system. We show that the radial flux is identically zero outside the event horizon, even though their wave functions are non-zero. Actually, we obtain the ingoing flux as well as the outgoing flux, but summing over these develops the zero flux near the event horizon exactly. This means that there is no leakage of information into the event horizon. Furthermore, the absorption cross section to the extremal CS-BTZ may vanish [14] and thus this extremal black hole may be transparent for any wave energy. Hence we argue that the extremal CS-BTZ black hole is a unitary system [11]. In this case, we cannot obtain discrete spectra like $\omega_e = -n/\ell$ because this belongs to the non-compact system.

On the other hand, the linearized equation for the massive graviton mode could not be represented by the massive scalar equation (14). The reason is that the massive scalar could

represent the case of the same highest weights of (h_L, h_R) with $h_L = h_R$. However, three graviton modes are massless left-moving, massless right-moving, and massive modes. They have the highest weights as $(2,0)$, $(0,2)$, and $(3/2 + \mu\ell/2, -1/2 + \mu\ell/2)$, respectively. The massive gravitons have negative energy for $\mu\ell > 1$. At the critical point of $\mu\ell = 1$, its highest weights reduce to $(2,0)$, those of a massless left-moving graviton. This corresponds to an unphysically propagating gauge boson. Considering conformal dimension $\Delta = h_L + h_R$, this boson may belong to the massive scalar with $\Delta = 2(m^2\ell^2 = 8)$. However, it has non-zero spin of $s = h_L - h_R = 2$. Hence, the quasinormal modes of graviton modes (gauge bosons) with spin could not be read off from the massive scalar modes with spin zero on the extremal CS-BTZ background. We propose to use the spin-dependent wave equation in Ref.[13], instead of Eq.(14).

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